Quantitative Aptitude

Trigonometry (Trigonometric Ratios and Identities)

Measurement of angle:

In circular system, unit of measurement of angle is radian. It is denoted by 1^c.

1 Radian: It is the angle subtended at the centre by the arc of length equal to radius of circle.

- **In circle, circumference = π × diameter**

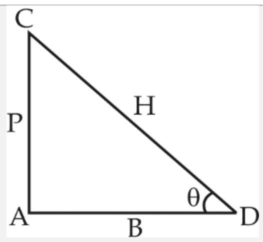
Where π = 3.1416 or $\frac{22}{7}$

- **Relationship between degree and radian: π radian = 180°**
- **When an arc subtends an angle θ radian at the centre of a circle of radius r then,**

$$\theta = \frac{\text{arc}}{\text{radius}}$$

- **1 radian = $\frac{180}{\left(\frac{22}{7}\right)}$ = 57° 16' 22"**
- **Thus, to change degree into radian, multiply by $\frac{\pi}{180°}$ and to change radian into degree multiply by $\frac{180°}{\pi}$**

Trigonometric Ratio:



In right angle triangle DAC,

There are six trigonometric ratios.

- $\sin\theta = \frac{P}{H}$ (P B P / H H B) $\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$
- $\cos\theta = \frac{B}{H}$ (P B P / H H B) $\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$
- $\tan\theta = \frac{P}{B}$ (P B P / H H B) $\tan\theta = \frac{\text{Perpendicular}}{\text{Base}}$
- $\cot\theta = \frac{B}{P}$ (P B P / H H B) $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$
- $\text{Sec}\,\theta = \frac{H}{B}$ (P B P / H H B) $\text{Sec}\,\theta = \frac{\text{Hypotenuse}}{\text{Base}}$
- $\text{cosec}\,\theta = \frac{H}{P}$ (P B P / H H B) $\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$

$\frac{\text{PBP}}{\text{HHB}} \rightarrow \frac{\text{Pandit Badri Prasad}}{\text{Har Har Bole}}$ (To memorize)

Trigonometric ratio of some specific angles

Trigonometric Ratios	0°	30°	45°	60°	90°	180°	270°	360°
	0rad.	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
cot θ	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	–1	∞	1
cosec θ	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	–1	∞

Some Important Result

- If sin x = 0 or tan x = 0, then x = nπ
- If cos x = 0 or cot x = 0, then x = $(2n + 1)\frac{\pi}{2}$
- Sin 15° = cos 75° = $\frac{\sqrt{3}-1}{2\sqrt{2}}$
- cos 15° = sin 75° = $\frac{\sqrt{3}+1}{2\sqrt{2}}$
- $\sin 22\frac{1°}{2} = \frac{\sqrt{2-\sqrt{2}}}{2}$
- $\cot 22\frac{1°}{2} = \frac{\sqrt{2+\sqrt{2}}}{2}$
- $\tan 22\frac{1°}{2} = \sqrt{2} - 1$
- $\cot 22\frac{1°}{2} = \sqrt{2} + 1$

- $\sin 18° = \cos 72° = \frac{\sqrt{5}-1}{4}$
- $\cos 18° = \sin 72° = \frac{\sqrt{10+2\sqrt{5}}}{4}$
- $\sin 36° = \cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4}$
- $\cos 36° = \sin 54° = \frac{\sqrt{5}+1}{4}$

Some important relations

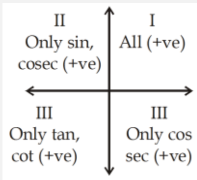
- $\sin\theta = \frac{1}{\text{cosec}\,\theta} \rightarrow \sin\theta.\text{cosec}\,\theta = 1$
- $\cos\theta = \frac{1}{\sec\theta} \rightarrow \sec\theta.\cos\theta = 1$
- $\tan\theta = \frac{1}{\cot\theta} \rightarrow \tan\theta.\cot\theta = 1$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Sign of Trigonometric functions in different Quadrants

We can obtain the sign of trigonometric function from Quadrant chart.

- We can remember as

Add	Sugar	To	coffee
↓	↓	↓	↓
All	sin/ cosec	tan/ cot	cos/ sec



Trigonometric ratio of Negative and Associated angle

Angle	– θ	(90 – θ)	(90 + θ)	(180 – θ)	(180 + θ)	(360 – θ)	(360 + θ)
sin θ	–sin θ	cos θ	cos θ	sin θ	– sin θ	– sin θ	sin θ
cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	cos θ	cos θ
tan θ	– tan θ	cot θ	– cot θ	– tan θ	tan θ	– tan θ	tan θ

From the above table we observe that only trigonometric ratio of (90 ± θ), and (270 ± θ) change and they change as.

$\sin \xleftrightarrow{\text{change}} \cos$

$\tan \xleftrightarrow{\text{change}} \cot$

$\text{cosec} \xleftrightarrow{\text{change}} \sec$

- Trigonometric ratio of (180±θ) and (360±θ) do not change
- Sign convention is used according to Quadrant.

Some results

when A + B = 90° i.e. sum of angles of acute angle triangle = 90°

- sin A = cos B
- tan A.tan B = 1
- cot A.cot B = 1

Basic Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta \Rightarrow \sec^2\theta - \tan^2\theta = 1$

$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1,$

If $\sec\theta + \tan\theta = P$, then, $\sec\theta - \tan\theta = \frac{1}{P}$

- $1 + \cot^2\theta = \text{cosec}^2\theta \Rightarrow \text{cosec}^2\theta - \cot^2\theta = 1$

$(\text{cosec}\,\theta + \cot\theta)(\text{cosec}\,\theta - \cot\theta) = 1,$

If $\text{cosec}\,\theta + \cot\theta = P$, then, $\text{cosec}\,\theta - \cot\theta = \frac{1}{P}$

Advanced trigonometric Identities

1. sin (A + B) = sin A cos B + cos A sin B
2. sin (A – B) = sin A cos B – cos A sin B
3. cos (A + B) = cos A cos B – sin A sin B
4. cos (A – B) = cos A cos B + sin A sin B
5. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

6. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
7. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
8. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
9. $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
10. $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
11. $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
12. $2\sin A \sin B = \cos(A-B) - \cos(A+B)$
13. $\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$
14. $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot A - \cot B}$
15. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
16. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
17. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
18. $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$
19. $\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$
20. $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
21. $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{1-\tan^2 A}{1+\tan^2 A}$
22. $\sin 3A = 3\sin A - 4\sin^3 A$
23. $\cos 3A = 4\cos^3 A - 3\cos A$
24. $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$
25. $\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = \frac{2\tan(A/2)}{1+\tan^2(A/2)}$
26. $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

$= 1 - 2\sin^2 A/2$

$= 2\cos^2 A/2 - 1$

27. $\tan A = \frac{2\tan(A/2)}{1-\tan^2(A/2)}$
28. $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A.\tan B - \tan B.\tan C - \tan A.\tan C}$

If A+B+C = 180°

- $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$
- $\frac{1}{\tan A.\tan B} + \frac{1}{\tan B.\tan C} + \frac{1}{\tan C.\tan A} = 1$
- $\cot A.\cot B + \cot B.\cot C + \cot C.\cot A = 1$

29. $\sin A \sin 2A.\sin 4A = \frac{1}{4}\sin 3A$
30. $\cos A \cos 2A.\cos 4A = \frac{1}{4}\cos 3A$
31. $\tan A \tan 2A \tan 4A = \tan 3A$
32. $\sin A \sin(60-A)\sin(60+A) = \frac{1}{4}\sin 3A$
33. $\cos A \cos(60-A)\sin(60+A) = \frac{1}{4}\cos 3A$
34. $\tan A \tan(60-A)\tan(60+A) = \tan 3A$

Trigonometry (Circular Measure of Angle)

Expression	Result
Sin (90° - θ)	cos θ
Cos (90° - θ)	sin θ
Tan (90° - θ)	cot θ
Cot (90° - θ)	tan θ
Sec (90° - θ)	csc θ
Csc (90° - θ)	sec θ
Sin (90° + θ)	cos θ

Cos (90° + θ)	-sin θ
Tan (90° + θ)	-cot θ
Cot (90° + θ)	-tan θ
Sec (90° + θ)	-csc θ
Csc (90° + θ)	sec θ
Sin (180° - θ)	sin θ
Cos (180° - θ)	-cos θ
Tan (180° - θ)	-tan θ
Sin (180° + θ)	-sin θ
Cos (180° + θ)	-cos θ
Example: tan 150°	Tan (90° + 60°) = -cot 60° = -1/
Example: sin 120°	Sin (90° + 30°) = cos 30° = √3/2
Example: cos 120°	Cos (180° - 60°) = -cos 60° = -1/2

Trigonometric Relations When A + B = 90°

S.No.	Condition / Identity	Explanation / Result
1	If A + B = 90°	A and B are complementary angles
2	sin A = cos B	sin A × sec B = 1
3	tan A = cot B	tan A × tan B = 1 or cot A × cot B = 1Example: tan 31° × tan 59° = 1
4	sec A = csc B	cos A × csc B = 1
5	$\sin^2 A + \sin^2 B = 1$	Equivalent to $\sin^2 A + \sin^2 (90° - A) = 1$
6	$\cos^2 A + \cos^2 B = 1$	Equivalent to $\sin^2 A + \cos^2 A = 1$

Trigonometry (Maximum and Minimum Value)

Range of trigonometric functions

S.No.	Trigonometric Ratio	Range
1	sin A	[−1, 1]
2	cos A	[−1, 1]
3	tan A	[−∞, ∞]
4	cot A	[−∞, ∞]
5	sec A	(−∞, −1]U[1, ∞)
6	cosec A	(−∞, −1]U[1, ∞)

$\sin^2 A$ and $\cos^2 A$ has minimum value 0 and maximum value 1 so range is (0, 1)

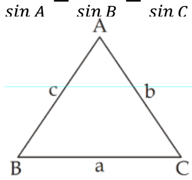
For values of 0° ≤ θ ≤ 90° then minimum and maximum value are given as

S. No.	Trigonometric Functions	Minimum value	Maximum value
1	$a\sin\theta + b\cos\theta$	$-\sqrt{a^2+b^2}$	$\sqrt{a^2+b^2}$
2	$a\sin^2\theta + b\,\mathrm{cosec}^2\theta$	$2\sqrt{ab}$	∞
3	$a\cos^2\theta + b\sec^2\theta$	$2\sqrt{ab}$	∞
4	$a\tan^2\theta + b\cot^2\theta$	$2\sqrt{ab}$	∞
5	$a\sin^2\theta + b\cos^2\theta$	a or b which ever is lowest	a or b which ever is greater
6	$a\sec^2\theta + b\,\mathrm{cosec}^2\theta$	$(\sqrt{a}+\sqrt{b})^2$	∞

Sine Rule:

In any ΔABC

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ = 2R, where R is circumradius.

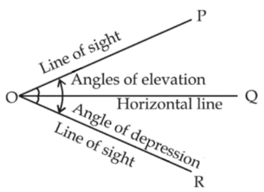
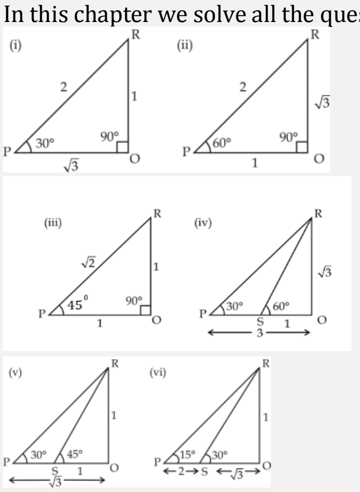


Cosine Rule:

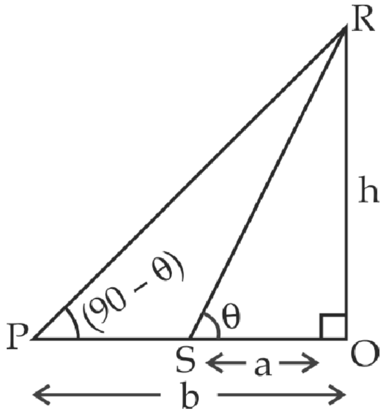
$a^2 = b^2 + c^2 - 2bc\cos A$, $b^2 = a^2 + c^2 - 2ac\cos B$, $c^2 = a^2 + b^2 - 2ab\cos C$

Important Results:	• sin θ + cos θ = a then, sin θ – cos θ = $\sqrt{2-a^2}$ • a sin θ + b cos θ = p and, a cos θ – b sin θ = q then, $a^2 + b^2 = p^2 + q^2$, q = $\pm\sqrt{a^2+b^2-p^2}$ • a sin θ + b cos θ = $\sqrt{a^2+b^2}$ then, sin θ = $\frac{a}{\sqrt{a^2+b^2}}$, cos θ = $\frac{b}{\sqrt{a^2+b^2}}$ • If a sec θ – b tan θ = $\sqrt{a^2+b^2}$ then, sec θ = $\frac{a}{\sqrt{a^2-b^2}}$, tan θ = $\frac{b}{\sqrt{a^2+b^2}}$ • If a cosec θ – b cot θ = $\sqrt{a^2-b^2}$ then, cosec θ = $\frac{a}{\sqrt{a^2-b^2}}$, cot θ = $\frac{b}{\sqrt{a^2+b^2}}$ • Some trigonometric function can be solved easily by considering it as algebraic function. **Example:** sin θ+ cosec θ = 2 It can be considered as x + $\frac{1}{x}$ where x = sin θ so, x = 1 and $\frac{1}{x}$ = 1, sin θ = 1 and $\frac{1}{cosec\ \theta}$ = 1

Trigonometry (Height and Distance)

Some Important terms  	**Line of Sight: The line of sight is the line drawn from the eye of an observer to the object.** **Angle of Elevation: When the object is above the horizontal level of our eye, we have to turn our head upwards to see an object. Here ∠POQ is the angle of elevation.** **Angle of Depression: When the object is below the horizontal level of our eye, we have to turn our head downwards to see an object. Here ∠QOR is the angle of depression.**
Some Important Points:	In this chapter we solve all the questions with the help of ratio. 

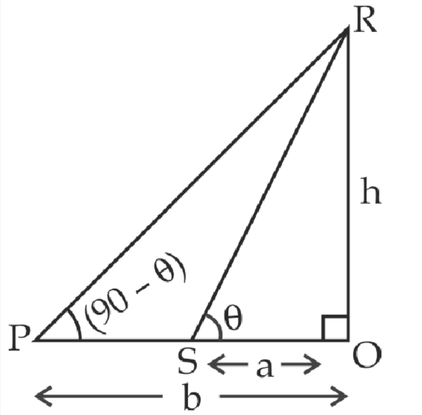
Concept 1:
If the angle of elevation of the top of a tower at two points which are at a distance of 'a' and 'b' metres from the foot of tower and on the same side of the tower are complementary. Then height of the tower is $\sqrt{ab}$.



Concept 2:
If a man wishes to find the height of a tower which stands on a horizontal plane. The angle of elevation of top of the tower is θ_1. On walking x units towards the tower. He find the angle of elevation becomes θ_2. Then the height of the tower is

$$h = \left[\frac{x \tan\theta_1 \tan\theta_2}{\tan\theta_2 - \tan\theta_1}\right]$$

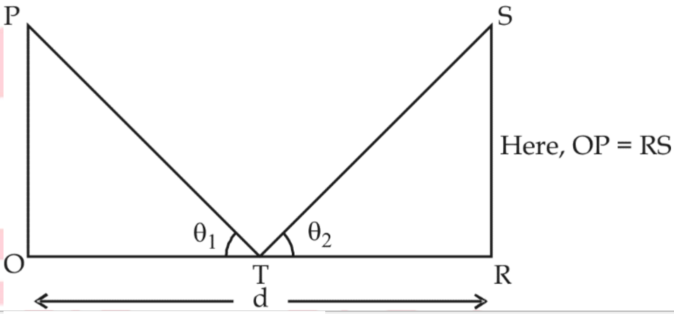
$$h = \frac{x}{\cot\theta_1 - \cot\theta_2}$$



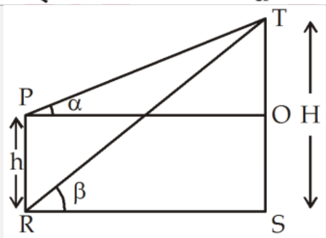
Concept 3:
If two poles of equal heights stand on either sides of a road which is 'd' units wide. At a point on the road between the poles, the elevation of the top of the poles are θ_1 and θ_2, then height of the poles is

$$h = \left[\frac{d \tan\theta_1 \tan\theta_2}{\tan\theta_1 + \tan\theta_2}\right]$$

$$h = \frac{d}{\cot\theta_1 + \cot\theta_2}$$



Concept 4:
From the top and bottom of the building of height 'h' units, the angle of elevation of the top of a tower are 'a' and 'b' respectively, then the height of the tower is $\left[\frac{h \tan\beta}{\tan\beta - \tan\alpha}\right]$



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